

THEOREMS ON TANGENCIES IN PROJECTIVE AND CONVEX GEOMETRY

ROLAND ABUAF

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Abstract

We discuss phenomena of tangency in Convex Optimization and Projective Geometry. Both theories have at disposal a powerful theory of duality. In both cases, the duality allows a nice interpretation of the contact locus of a hyperplane with an embedded variety. In this paper, we investigate more precisely some similarities between the theorems on tangencies existing in both theories. We focus in particular on a theorem of Anderson and Klee and its conjectural reformulation in Algebraic Geometry. If true, this conjecture would have significant consequences for Projective Geometry.

1 Introduction

Let $X \subset \mathbb{R}^n$ be a compact convex body whose interior contains 0. There have been considerable efforts to classify the *singularities* of the points lying in the boundary of X . A clear picture of the situation was probably given for the first time by Anderson and Klee [AK52].

Definition 1.0.1 *Let $X \subset \mathbb{R}^n$ be a compact convex body whose interior contains 0, let $X^* \subset \mathbb{R}^{n*}$ be its dual body and let $x \in \partial X$. We say that x is an **r -singular point** of X if the **exposed face** of X^* relative to x^\perp has dimension at least r .*

Theorem 1.0.2 *Let $X \subset \mathbb{R}^n$ be a compact convex body whose interior contains 0 and let $r \in \{0, \dots, n-1\}$. The set of r -singular points of X can be covered by countably many compact subsets of finite $n-r-1$ -dimensional Hausdorff measure.*

Now we turn to a similar situation in Projective Geometry. Let $X \subset \mathbb{P}_{\mathbb{C}}^n$ be an irreducible, non-degenerate projective variety. Zak found a bound for the dimension of the contact locus of any linear space with X [Zak93].

Theorem 1.0.3 *Let $X \subset \mathbb{P}_{\mathbb{C}}^n$ be an irreducible, non degenerate projective variety and let $L \subset \mathbb{P}_{\mathbb{C}}^N$ be a linear space. Denote by $X_L = \{x \in X, T_{X,x} \subset L\}$, we have the inequality:*

$$\dim(X_L) \leq \dim(L) - \dim(X) + b + 1,$$

where $b = \dim X_{\text{sing}}$.

Both theorems tell us that *support loci* are subject to dimensional constraints. However, theorem 1.0.2 bounds the dimension of a family of hyperplanes when the dimension of the contact locus of the general member is known, whereas theorem 1.0.3 bounds the dimension of the contact locus of a single hyperplane. An Algebro-Geometric statement similar to theorem 1.0.2 was used in the first version of [RS10]. Unfortunately, no convincing proof was given there.

Conjecture 1.0.4 *Let $X \subset \mathbb{P}_{\mathbb{C}}^n$ be a non-degenerate, irreducible projective variety and let $X^* \subset \mathbb{P}_{\mathbb{C}}^N$ be its projective dual. Let $r \in \{0, \dots, n-1\}$ and denote by $X^*\langle r \rangle = \{H^\perp \in X^*, \dim \langle X_H \rangle \geq r\}$, where $\langle . \rangle$ denotes the scheme-theoretic linear span and X_H is the tangency locus of H with X . We have the inequality:*

$$\dim X^*\langle r \rangle \leq n - r - 1.$$

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2 Dualities and Contact Loci

2.1 A Common Setting for the Dualities

Here we formulate, in a common language, the duality for convex bodies and for projective varieties. In the following, the space \mathbb{E}^n either denotes the complex projective space $\mathbb{P}_{\mathbb{C}}^n$ or the real euclidean space \mathbb{R}^n . An object $X \subset \mathbb{E}^n$ refers to a compact convex body in \mathbb{R}^n whose interior contains 0 or to a reduced (irreducible) projective scheme in $\mathbb{P}_{\mathbb{C}}^n$. If X is a convex body, then ∂X is the boundary of X . If X is a reduced projective scheme, then, by convention, $\partial X = X$. We denote by \overline{Z} the convex hull or the Zariski closure of an object $Z \subset \mathbb{E}^n$.

Definition 2.1.1 *Let $X \subset \mathbb{R}^n$ be a compact convex body whose interior contains 0, let $y \in X$ and let $H \subset \mathbb{R}^n$ be a hyperplane. We say that H has **contact** with X at y , if for all $x \in X$ we have $\langle H^\perp, x \rangle \leq 1$, and $\langle H^\perp, y \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the evaluation pairing between \mathbb{R}^{n*} and \mathbb{R}^n .*

Note that if H has contact with X at y , then necessarily $y \in \partial X$.

Definition 2.1.2 *Let $X \subset \mathbb{P}_{\mathbb{C}}^n$ be a reduced (irreducible) projective scheme and let $H \subset \mathbb{P}_{\mathbb{C}}^n$ be a hyperplane.*

*Let $y \in X_{\text{smooth}}$. We say that H has **contact** with X at y if $T_{X,y} \subset H$.*

*Let $y \in X_{\text{sing}}$. We say that H has **contact** with X at y if there exist sequences $(y_m) \in X_{\text{smooth}}$ and $(H_m^\perp) \in \mathbb{P}_{\mathbb{C}}^{n*}$ such that $H^\perp = \lim H_m^\perp$, $y = \lim y_m$ and H_m has contact with X at y_m for all $m \in \mathbb{N}$.*

Now we can state both dualities in a common setting.

Theorem 2.1.3 (Duality) *Let $X \subset \mathbb{E}^n$ be an object. Consider the incidence $I_X = \{(H^\perp, x) \in \mathbb{E}^{n*} \times \partial X, H \text{ has contact with } X \text{ at } x\}$, and the natural diagram:*

$$\begin{array}{ccc} & I_X & \\ q \swarrow & & \searrow p \\ \mathbb{E}^{n*} \supset \partial(X^*) & & \partial X \subset \mathbb{E}^n \end{array}$$

Figure 1: conormal diagram

Let $X^ = \overline{q(I_X)}$. We have $I_{X^*} = I_X$. As a consequence, we have $X^{**} = X$ and $q(I_X) = \partial(X^*)$.*

Note that if $X \subset \mathbb{E}^n$ is a reduced projective scheme, then $q(I_X)$ is obviously Zariski closed. In this case, the equality $q(I_X) = \partial(X^*)$ is trivial since, in our notations, $\partial X^* = X^*$. Note also that, by construction, for all $H^\perp \in q(I_X)$, we have:

$$x \in p(q^{-1}(H^\perp)) \Leftrightarrow H \text{ has contact with } X \text{ at } x.$$

The object $p(q^{-1}(H^\perp))$ is called the **contact locus** of H along X . In Convex Geometry, the set $p(q^{-1}(H^\perp))$ is often called the *exposed face* of X relative to H , while in Projective Geometry it is known as the *tangency locus* of H along X . The duality says that the set of hyperplanes which have contact with X at x is equal to the contact locus of x^\perp along X^* . That is, for all $x \in X$, we have:

$$H^\perp \in q(p^{-1}(x)) \Leftrightarrow x^\perp \text{ has contact with } X^* \text{ at } H^\perp.$$

A proof of this result can be given in the framework of Lagrangian Geometry. Indeed the notion of *contact* allows one to define in a uniform way Lagrangian manifolds for Projective and Convex Geometry.

2.2 The Principle of Anderson and Klee

In this section, we formulate the principle of Anderson and Klee in a common setting for Projective Geometry and Convex Geometry.

Notations 2.2.1 *Let $X \subset \mathbb{E}^n$ be an object. The **linear span** of X , which we denote by $\langle X \rangle$ is the smallest linear subspace of \mathbb{E}^n which contains X .*

In the case $Z \subset \mathbb{E}^n$ is a non-reduced scheme, the subspace $\langle Z \rangle$ is the scheme-theoretic linear span of Z .

Definition 2.2.2 *Let $X \subset \mathbb{E}^n$ be an object. A point $x \in X$ is said to be a r -singular point in X if $\dim \langle q(p^{-1}(x)) \rangle \geq r$. The set of r -singular points of X is denoted by $X \langle r \rangle$.*

The following result is the archetype of the theorem on tangencies which should be true in all geometries. It was proven by Anderson and Klee (see [AK52], or [Sch93] for a modern presentation) in the context of Convex Geometry and was used in the first version of [RS10] in the context of Projective Geometry.

Conjecture 2.2.3 *Let $X \subset \mathbb{E}^n$ be an object, we have the inequality:*

$$\dim X \langle r \rangle \leq n - r - 1.$$

Here the dimension must be understood as the Hausdorff dimension or the algebraic dimension, depending on the theory. Note that this conjecture is of cohomological nature in Algebraic Geometry. A similar statement can be formulated as follows.

Conjecture 2.2.4 *Let $X \subset \mathbb{P}^n$ be a smooth, irreducible, non-degenerate projective variety. Denote by $X^* \langle r \rangle$ the set:*

$$X^* \langle r \rangle = \{H^\perp \in X^*, \dim H^0(\mathcal{J}_{p(q^{-1}(H^\perp))}(1)) \leq n - r\}.$$

We have the inequality:

$$\dim X^* \langle r \rangle \leq n - r - 1,$$

for all $r \in \{0, \dots, n\}$.

Here $\mathcal{J}_{p(q^{-1}(H^\perp))}$ denotes the ideal sheaf of $p(q^{-1}(H^\perp))$ in \mathbb{P}^n , where p, q are defined in the conormal diagram. One expects that similar bounds could be found for the dimension of the set of points $H^\perp \in X^*$ such that $\dim H^0(\mathcal{J}_{p(q^{-1}(H^\perp))}(k))$ is "small" enough.

Using the theory developped by Hironaka around the notion of normal flatness [Hir64] and a result of Lê-Teissier [LT88], one can prove the following result in Projective Geometry.

Proposition 2.2.5 *Let $X \subset \mathbb{P}^n$ be an irreducible, reduced, non-degenerate projective variety. Let*

$$\tilde{X}^* \langle r \rangle = \{H^\perp \in X^*, \dim H^0(\mathcal{J}_{|p(q^{-1}(H^\perp))|_{red}}(1)) \leq n - r\}.$$

We have the inequality:

$$\dim \tilde{X}^* \langle r \rangle \leq n - r - 1.$$

Here $|p(q^{-1}(H^\perp))|_{red}$ is the reduced space underlying $p(q^{-1}(H^\perp))$.

3 Applications to Projective Geometry

If true, conjecture 2.2.4 would have significant consequences for Projective Geometry. In fact, even proposition 2.2.5 can be used to prove a generalization of Severi's theorem.

Notations 3.0.6 *Let $X \subset \mathbb{P}_{\mathbb{C}}^n$ be an irreducible projective variety. We denote by $X^*(r)$ the set $X^*(r) = \{H \in X^*, \dim p(q^{-1}(H)) \geq r\}$.*

Theorem 3.0.7 *Let $X \subset \mathbb{P}_{\mathbb{C}}^5$ be a smooth, irreducible, non-degenerate projective surface and let $X^* \subset \mathbb{P}^{5*}$ its projective dual. We have $\dim X^*(1) \leq 2$, with equality if and only if X is the Veronese surface.*

Sketch of the proof:

► By assumption $X \neq \mathbb{P}^2$, so X does not contain a 2-dimensional family of lines. As a consequence of proposition 2.2.5, we see that $\dim X^*(1) \leq 2$.

Assume that $\dim X^*(1) = 2$, proposition 2.2.5 again shows that for all $H^\perp \in X^*(1)$, the curve-components of $(H \cap X)_{\text{sing}}$ are plane curves.

Let $H^\perp \in X^*(1)$ be a general point and let k be the maximum of the degree of the curve-components of $|(H \cap X)_{\text{sing}}|_{\text{red}}$. Assume that $k \geq 3$. Then, there is a plane curve, say C , in $|(H \cap X)_{\text{sing}}|_{\text{red}}$ such that all lines in $\langle C \rangle$ are trisecants to X . But this is true for general $H^\perp \in X^*(1)$, so that a careful count of dimension shows that we have a 4-dimensional family of trisecants to X . This is impossible by the trisecants lemma.

As a consequence, the smooth surface X is covered by a 2-dimensional family of conics, it is the Veronese surface. ◀

Note that theorem 3.0.7 obviously implies Severi's original result. Indeed, if $X \subset \mathbb{P}_{\mathbb{C}}^5$ is a smooth, irreducible, non-degenerate surface whose secant variety does not cover the ambient space, then Terracini's lemma implies that $\dim X^*(1) = 2$. Another proof of Severi's result, relying on similar techniques as the above ones, is due to Zak and is a consequence of theorem 1.0.3. Hence, one may hope that theorem 1.0.3 and conjecture 2.2.4 could be considered in a common setting. As such, these results are perhaps incarnations of a deeper principle, which has yet to be discovered.

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